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Equations governing steady three-dimensional large-amplitude motion of a stratified fluid

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The exact equations governing three-dimensional motion of an inviscid non-diffusive incompressible fluid stratified in density or of an inviscid non-diffusive gas stratified in entropy are given and briefly discussed.

1. Introduction

Since the Reynolds number and the Péclet number for most flows occurring in the atmosphere are extremely large, the study of the flows of an inviscid and non-diffusive fluid is relevant to the understanding of many atmospheric phenomena. In this paper inviscidness and non-diffusiveness are assumed throughout for the fluids considered. The equations governing steady two-dimensional flows were given by Madame Dubreil-Jacotin for an incompressible fluid of variable density (1935, p. 345, equation (B)) and for an ideal gas of variable entropy (1935, p. 346, equation (*b*)). These equations were later rediscovered by Prof. Long (1953*a*, *b*), and have been effectively and fruitfully utilized by him in his excellent studies of atmospheric waves. Simplified versions of these equations were given by Yih (1958, 1960*a*, *b*), who used modified stream functions to remove the non-linear terms arising from the convective terms in the Eulerian expression of the acceleration in the equations of motion.

For three-dimensional motion the corresponding equations are not available to this day. Of course we always have the Euler equations of motion, the equation of continuity, and the equation of conservation of density or of entropy. But these basic equations, four of which non-linear, do not give much promise. Before their solution can be attempted, their number need to be reduced by some sort of elimination, in the same way that such a reduction is effected to produce the final single equation of Dubreil-Jacotin for either an incompressible fluid or a gas. In this paper we shall present the three-dimensional counterparts of the equations of Madame Dubreil-Jacotin.

2. Some preliminaries

It is very helpful to visualize the very pronounced properties of stratified flows of inviscid and non-diffusive fluids before we attempt to derive the equations sought. If we assume the motion to be steady, the conservation equations are

$$(\mathbf{v}\,.\,\mathrm{grad})\,\rho = 0 \quad \text{or} \quad (\mathbf{v}\,.\,\mathrm{grad})\,S = 0, \tag{1}$$

in which $\mathbf{v}$ is the velocity vector, ρ the density, and S the entropy. These equations show that in steady flows the velocity vector must lie in isopycnic surfaces or surfaces of constant entropy. Since the circulation along any circuit lying entirely in an isopycnic surface or a surface of constant entropy is preserved, this circulation must be zero if the flow is supposed to have been established from rest. Hence for such a flow the vorticity vector must also lie in an isopycnic surface or a surface of constant entropy. (See Yih 1965, p. 14.) Then for the case of an incompressible fluid

$$\mathbf{v} = \operatorname{grad} a \times \operatorname{grad} \rho, \quad \boldsymbol{\omega} = \operatorname{grad} b \times \operatorname{grad} \rho, \tag{2 a, b}$$

in which $\boldsymbol{\omega}$ is the vorticity vector, a is a stream function (Yih 1957), the other stream function being ρ, and b a vorticity function of Clebsch (Lamb 1945, p. 248), the other vorticity function being ρ. Of course $\mathbf{v}$ and $\boldsymbol{\omega}$ are related by

$$\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}. \tag{3}$$

For a compressible fluid (2 a) and (2 b) are to be replaced by

$$\rho\mathbf{v} = \operatorname{grad} a \times \operatorname{grad} S \quad \text{and} \quad \boldsymbol{\omega} = \operatorname{grad} b \times \operatorname{grad} S. \tag{4 a, b}$$

3. Derivation of the equations for an incompressible fluid

The equations of steady motion are

$$(\rho\mathbf{v} . \operatorname{grad})\,\mathbf{v} = -\operatorname{grad} p + \rho\mathbf{F}, \tag{5}$$

in which p is the pressure and $\mathbf{F}$ the body force per unit mass given by

$$\mathbf{F} = -\operatorname{grad} gz,$$

z being the Cartesian co-ordinate measured in the direction of the vertical. If we assume (Yih 1958)

$$\mathbf{v}' = (\rho/\rho_0)^{\frac{1}{2}}\,\mathbf{v}, \quad \boldsymbol{\omega}' = \operatorname{curl} \mathbf{v}', \tag{6}$$

(4) can be written as

$$(\rho_0 \mathbf{v}' . \nabla)\,\mathbf{v}' = -\operatorname{grad} p + \rho\mathbf{F}, \tag{5a}$$

which can further be written as

$$-\rho_0 \mathbf{v}' \times \boldsymbol{\omega}' = -\operatorname{grad} \chi + \rho\mathbf{F}, \tag{7}$$

with

$$\chi = p + \tfrac{1}{2}\rho(u^2 + v^2 + w^2),$$

u, v, and w being the components of $\mathbf{v}$ in the directions of increasing Cartesian co-ordinates x, y, and z respectively, and ρ_0 being a reference density.

Now since $\mathbf{v}'$ and $\boldsymbol{\omega}'$ are still solenoidal, we can write

$$\mathbf{v}' = \operatorname{grad} \alpha \times \operatorname{grad} \rho, \quad \boldsymbol{\omega}' = \operatorname{grad} \beta \times \operatorname{grad} \rho. \tag{8 a, b}$$

Then a simple calculation shows that

$$\mathbf{v}' \times \boldsymbol{\omega}' = -(\mathbf{v}' . \operatorname{grad} \beta) \operatorname{grad} \rho = (\boldsymbol{\omega}' . \operatorname{grad} \alpha) \operatorname{grad} \rho. \tag{9}$$

This is as it should be, since both the velocity vector and the vorticity vector lie in a surface of constant ρ, and thus their vector product must be parallel to $\operatorname{grad} \rho$. Substituting (9) in (7), separating the result into three equations, multiplying these respectively by dx, dy and dz, and adding, we obtain, after an obvious simplification,

$$\rho_0 \boldsymbol{\omega}' . \mathrm{grad}\, \alpha = (dH/d\rho) - gz, \tag{10}$$

in which

$$H = \chi + \rho g z \tag{11}$$

is a function of ρ only, since the Bernoulli function is constant in an isopycnic surface for steady flows. Now since $\boldsymbol{\omega}'$ is in an isopycnic surface,

$$\boldsymbol{\omega}' . \mathrm{grad}\, \rho = 0. \tag{12}$$

We shall now use the second equation in (6) and (8 a) to obtain

$$\left.\begin{aligned} \xi' &= (\rho_{yy}+\rho_{zz})\,\alpha_x - \rho_{xy}\,\alpha_y - \rho_{xz}\alpha_z - (\alpha_{yy}+\alpha_{zz})\,\rho_x + \alpha_{xy}\,\rho_y + \alpha_{xz}\,\rho_z, \\ \eta' &= (\rho_{zz}+\rho_{xx})\,\alpha_y - \rho_{yx}\,\alpha_x - \rho_{yz}\alpha_z - (\alpha_{zz}+\alpha_{xx})\,\rho_y + \alpha_{yx}\,\rho_x + \alpha_{yz}\,\rho_z, \\ \zeta' &= (\rho_{xx}+\rho_{yy})\,\alpha_z - \rho_{zx}\alpha_x - \rho_{zy}\,\alpha_y - (\alpha_{xx}+\alpha_{yy})\,\rho_z + \alpha_{zx}\,\rho_x + \alpha_{zy}\,\rho_y, \end{aligned}\right\} \tag{13}$$

in which ξ', η' and ζ' are the three components of $\boldsymbol{\omega}'$. With (13) introduced in (10) and (12), we obtain two equations of the two unknowns α and ρ, which are the equations sought. For two-dimensional flows in the (x, z)-plane, $\boldsymbol{\omega}'$ has only the component η' and $\mathrm{grad}\,\rho$ in the (x, z)-plane. Hence (12) is automatically satisfied. Furthermore, if we keep the dimensions correct and write

$$\alpha = (Vd/\rho_0)\, y,$$

in which V is a reference velocity, d a reference length, and ρ_0 a reference density, we have

$$\rho_{xx} + \rho_{zz} = \frac{\rho_0}{(Vd)^2}\left(\frac{dH}{d\rho} - gz\right). \tag{14}$$

With

$$\xi = \frac{x}{d}, \quad \zeta = \frac{z}{d}, \quad r = \frac{\rho}{\rho_0}, \quad h = \frac{H}{\rho_0 V^2}, \quad F^2 = \frac{V^2}{gd},$$

(14) becomes

$$r_{\xi\xi} + r_{\zeta\zeta} = (dh/dr) - F^{-2}\zeta. \tag{15}$$

We can also recover Yih's form of the equation of Dubreil-Jacotin if we put

$$\alpha = (d\psi'/d\rho)\, y,$$

in which ψ' is the modified stream function used by Yih and

$$u' = \psi'_z, \quad w' = -\psi'_x.$$

Since neither ψ' nor ρ depends on y, the result is

$$\psi'_{xx} + \psi'_{zz} = \frac{1}{\rho_0}\frac{dH}{d\psi'} - \frac{gz}{\rho_0}\frac{d\rho}{d\psi'}, \tag{16}$$

which is the equation of Dubreil-Jacotin in Yih's form.

Equations (10) and (12), with $\boldsymbol{\omega}'$ given by (13), are integrated forms of the equations of motion. That (10) results from integration is obvious from its derivation. Even (12) results from an integration—the integration along any closed circuit in an isopycnic line to obtain the circulation, which is zero if the motion started from rest. Thus (10) and (12) are several steps in advance of the Euler equations of motion. The equation of continuity is automatically satisfied by $(2a)$ or $(8a)$. The first equation in (1), which has been used to obtain $(5a)$, is also automatically satisfied by $(2a)$, or $(8a)$ taken together with the definition of v' in (6).

The left-hand side of (10) is linear in ρ, and the left-hand side of (12) is linear in α. In this sense the left-hand sides of (10) and (12) are quasi-linear. If $dH/d\rho$ is linear in ρ, (10) and (12) are quasi-linear, and a calculation can be performed by first assuming a plausible form for α, solving (10) for ρ, using the result in (12) and solving it for α, and repeating the process.

4. Derivation of the equations for a compressible fluid

For a compressible fluid in steady motion the basic equations are still (5). The equation of continuity

$$\operatorname{div}(\rho\mathbf{v}) = 0 \tag{17}$$

is automatically satisfied by $(4a)$. We shall again use the variable λ defined by

$$\lambda = \frac{\rho}{\rho_0}\left(\frac{p_0}{p}\right)^{1/\gamma} = \text{constant} \times e^{-S/c_p}, \tag{18}$$

in which ρ_0 is a reference density and p_0 a reference pressure, and γ is the ratio of the specific heat c_p at constant pressure to the specific heat c_v at constant volume. With the substitutions (Yih 1960b)

$$\mathbf{v}' = \sqrt{\lambda}\,\mathbf{v}, \quad \rho' = \rho/\lambda, \quad p' = p, \tag{19}$$

the basic equations of motion can be written as

$$(\rho'\mathbf{v}'.\nabla)\,\mathbf{v}' = -\operatorname{grad} p' + \rho'\lambda\mathbf{F}, \tag{20}$$

in which

$$\rho'/(p')^{1/\gamma} = \text{constant}. \tag{21}$$

Equation (20) can be written as

$$-\mathbf{v}' \times \boldsymbol{\omega}' = -\operatorname{grad}\chi + \lambda\mathbf{F}, \tag{22}$$

in which

$$\chi = \int \frac{dp'}{\rho'} + \frac{q'^2}{2}, \tag{23}$$

q' being the magnitude of $\mathbf{v}'$. The equation of continuity in terms of ρ' and $\mathbf{v}'$ is, in virtue of the second equation in (1),

$$\operatorname{div}(\rho'\mathbf{v}') = 0,$$

which is automatically satisfied by

$$\rho'\mathbf{v}' = \operatorname{grad}\alpha \times \operatorname{grad}\lambda. \tag{24}$$

We use this form not only because the satisfaction of the equation of continuity is assured, but also because the velocity vectors must lie in surfaces of constant S or λ. Similarly, since the vorticity vector $\boldsymbol{\omega}$ (and hence $\boldsymbol{\omega}'$) must also lie in surfaces of constant λ,

$$\boldsymbol{\omega}' = \operatorname{grad}\beta \times \operatorname{grad}\lambda. \tag{25}$$

A simple calculation shows that

$$\rho'\mathbf{v}' \times \boldsymbol{\omega}' = -(\rho'\mathbf{v}'.\operatorname{grad}\beta)\operatorname{grad}\lambda = (\boldsymbol{\omega}'.\operatorname{grad}\alpha)\operatorname{grad}\lambda. \tag{26}$$

Substituting this in (22), multiplying it by $d\mathbf{x}$, and integrating, we obtain

$$\frac{1}{\rho'}\boldsymbol{\omega}'.\operatorname{grad}\alpha = \frac{dH}{d\lambda}-gz, \tag{27}$$

in which

$$H = \chi+gz\lambda. \tag{28}$$

The other equation is

$$\boldsymbol{\omega}'.\operatorname{grad}\lambda = 0. \tag{29}$$

The expression for $\boldsymbol{\omega}'$ is now a little more complicated. It is

$$\boldsymbol{\omega}' = \operatorname{curl}\mathbf{v}' = \operatorname{curl}[(1/\rho')\operatorname{grad}\alpha\times\operatorname{grad}\lambda]. \tag{30}$$

We shall not expand it in full. Equations (27) and (28), with $\boldsymbol{\omega}'$ given by (30), are the equations sought. The quantity ρ' can be expressed in terms of H, q' and λ by the use of (28).

For two-dimensional flows,

$$\alpha = \rho_0 Vdy,$$

and (27) becomes

$$\left(\frac{\rho_0 Vd}{\rho'}\right)^2\left[(\lambda_{xx}+\lambda_{zz})-\frac{1}{\rho'}(\lambda_x\rho'_x+\lambda_z\rho'_z)\right] = \frac{dH}{d\lambda}-qz. \tag{31}$$

If we put

$$\alpha = (d\psi'/d\lambda)\,y, \tag{32}$$

we obtain Yih's form of the equation of Dubreil-Jacotin

$$\nabla^2\psi'-\frac{1}{\rho'}(\psi'_x\rho'_x+\psi'_z\rho'_z)-gz\rho'^2\frac{d\lambda}{d\psi'} = \rho'^2\frac{dH}{d\psi'}, \tag{33}$$

in which

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}.$$

Equations (27) and (28) are a few steps in advance of the basic equations of motion because they have been obtained from the latter equations by integration.

5. Discussion

Actually, (10) and (12) governing the motion of a stratified incompressible fluid are also the equations governing steady vortex motion of a homogeneous incompressible fluid. Isopycnic surfaces would of course have no definite meaning, but we can replace ρ by L, which is constant on a Lamb surface with streamlines and vorticity lines imbedded in it. Since ρ is now constant, the last term in (10) drops out for vortex motion, which is then governed by

$$\rho\boldsymbol{\omega}.\operatorname{grad}\alpha = \frac{dH}{dL} \quad\text{and}\quad \boldsymbol{\omega}.\operatorname{grad}L = 0, \tag{34}$$

with $\boldsymbol{\omega}$ given by (13), in which ρ is replaced by L, and the accents on ξ', η' and ζ' are removed. Of course, the motion is now not assumed to have started from rest. It would be irrotational in that case.

For a homentropic gas in steady vortex motion, the governing equations are

$$\frac{1}{\rho}\boldsymbol{\omega}.\operatorname{grad}\alpha = \frac{dH}{dL} \quad \text{and} \quad \boldsymbol{\omega}.\operatorname{grad} L = 0, \tag{35}$$

in which

$$\boldsymbol{\omega} = \operatorname{curl}\mathbf{v} = \operatorname{curl}[(1/\rho)\operatorname{grad}\alpha \times \operatorname{grad} L]. \tag{36}$$

Equations (34) and (35) are the results of first integrations of the vorticity equations. It is a little surprising that they have not been found before.

Finally, we remark that the solution of (10) and (12), or (27) and (29), or (34), or (35), is not unique. For if α is a solution so is $\alpha + F(\rho)$, or $\alpha + F(\lambda)$, or $\alpha + F(L)$, as the case may be, but the velocity field is uniquely determined.

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